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# TENSOR NORMS FOR QUANTUM INFORMATION THEORY

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| **Period** | 6 months beginning not later than:<br>☐ January ☐ February ☐ March ☐ April ☐ May ☒ June ☐ July ☐ September 2021 |
| **Internship supervisor(s)** | name: Ion Nechita<br>e-mail: ion.nechita@univ-tlse3.fr<br>group: LPT - QUANTWARE |
| **Location** | LPT, bat. 3R1, IRSAMC, Université Paul Sabatier<br>118, route de Narbonne<br>31062 Toulouse France |
| This research master's degree research project could be followed by a PhD | ☒ YES ☐ NO |

Given a set of m normed vector spaces $X_1, \ldots, X_m$, a norm on $X = X_1 \otimes \cdots \otimes X_m$ is called a *tensor norm* (or a cross norm) if, for all simple tensors $x = x_1 \otimes \cdots \otimes x_m$, we have

$$\|x\| = \|x_1\|_1 \cdots \|x_m\|_m$$

It has been known since the pioneering work of Grothendieck [Gro56] that there is a smallest and a largest such norms, namely the *injective* and the *projective* tensor norms

$$\|x\|_\varepsilon := \sup\{\langle \varphi_1 \otimes \cdots \otimes \varphi_m, x\rangle \ : \ \varphi_i \in X_i^*, \|\varphi_i\|_i^* \leq 1\}$$

$$\|x\|_\pi := \inf\left\{\sum_i \|x_1^{(i)}\|_1 \cdots \|x_m^{(i)}\|_m \ : \ x = \sum_i x_1^{(i)} \otimes \cdots \otimes x_m^{(i)}\right\}.$$

For example, when identifying $\mathbb{C}^d \otimes \mathbb{C}^d$ with the space of d × d matrices, the injective norm for the $L_2$ norms on $\mathbb{C}^d$ corresponds to the operator norm, while the projective norm corresponds to the Schatten 1-norm (or nuclear norm). For more than two tensor factors, it is known that it is NP-hard to compute these norms [HL13, FL18]. In quantum information theory, most of the time the relevant Banach space is the Schatten 1 space ($M_d(\mathbb{C})$, trace norm), see [AS17]. It is a folklore result that a quantum state $\rho \in M_{d1} \otimes \cdots \otimes M_{dm}$ is separable if $||\rho||_\pi = \mathrm{Tr}\,\rho = 1$, where the projective norm is relative to the Schatten 1-norms (or trace norms) on the tensor factors. Moreover, in the case of pure states, the injective norm of a tensor is known in quantum information theory as the geometric measure of entanglement [WG03].

The candidate will study several recent proposals [XQLJ18, MS18] for algorithms approximating the projective (or nuclear) norm, with goal of applying such results to the problem of quantum entanglement [HHHH09]. Working in different scenarios (pure states, symmetric states, low rank states, general mixed states), the different methods
for computing the tensor nuclear norm will be analyzed from the point of view of entanglement/separability guarantees, and computational complexity (to be compared with that of traditional methods, such as sum-of-squares hierarchies [DPS04] and ε-nets). The use of different tensor decompositions [KB09] in quantum information theory is a different topic which will be explored.

References:

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[MS18] Andrea Montanari and Nike Sun. Spectral algorithms for tensor completion. Communications on Pure and Applied Mathematics, 71(11):2381–2425, 2018.
[WG03] Tzu-Chieh Wei and Paul M Goldbart. Geometric measure of entanglement and applications to bipartite and multipartite quantum states. Physical Review A, 68(4):042307, 2003.
[XQLJ18] Shengke Xue, Wenyuan Qiu, Fan Liu, and Xinyu Jin. Low-rank tensor completion by truncated nuclear norm regularization. In 2018 24th International Conference on Pattern Recognition (ICPR), pages 2600–2605. IEEE, 2018.

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| **Keywords, areas of expertise** | Quantum information theory, quantum computing, quantum entanglement, tensor norms, multilinear algebra |
| **Required skills for the internship** | The candidate should have a strong mathematical profile. Competences in linear and multilinear algebra and quantum (information) theory are the most important for the research project (but not strictly required). |